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Growth rates of amenable groups

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Abstract. Let F_m be a free group with m generators and let R be a normal subgroup such that F_m/R projects onto $\mathbb{Z}$. We give a lower bound for the growth rate of the group F_m/R' (where R' is the derived subgroup of R) in terms of the length $\rho = \rho(R)$ of the shortest non-trivial relation in R. It follows that the growth rate of F_m/R' approaches $2m - 1$ as ρ approaches infinity. This implies that the growth rate of an m-generated amenable group can be arbitrarily close to the maximum value $2m - 1$. This answers an open question of P. de la Harpe. We prove that such groups can be found in the class of abelian-by-nilpotent groups as well as in the class of virtually metabelian groups.

1 Introduction

Let G be a finitely generated group and A a fixed finite set of generators for G. We denote by $l(g)$ the *word length* of an element $g \in G$ in the generators A, i.e. the length of a shortest word in the alphabet $A^{\pm 1}$ representing g. Let $B(n)$ denote the ball $\{g \in G \mid l(g) \leqslant n\}$ of radius n in G with respect to A. The *growth rate* of the pair (G, A) is the limit

$$\omega(G, A) = \lim_{n\to\infty} \sqrt[n]{|B(n)|}.$$

(Here $|X|$ denotes the number of elements of a finite set X.) This limit exists due to the submultiplicativity property of the function $|B(n)|$; see for example [5, VI.C, Proposition 56]. Clearly $\omega(G, A) \geqslant 1$. A finitely generated group G is said to be of *exponential growth* if $\omega(G, A) > 1$ for some (and hence in fact for any) finite generating set A. Groups with $\omega(G, A) = 1$ are groups of *subexponential* growth.

Let $|A| = m$. It is known that $\omega(G, A) = 2m - 1$ if and only if G is freely generated by A; see [3, Section V]. In this case G is non-amenable whenever $m > 1$.

A finitely generated group which is non-amenable is necessarily of exponential growth [1]. The following interesting question is due to P. de la Harpe.

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Question (see [5, VI.C, 62]). For an integer $m \geqslant 2$, does there exist a constant c_m with $1 < c_m < 2m - 1$, such that G is not amenable whenever $\omega(G, A) \geqslant c_m$?

We show that the answer to this question is negative. Thus, given $m \geqslant 2$, there exists an amenable group on m generators with growth rate as close to $2m - 1$ as one likes.

It is worth noticing that for every $m \geqslant 2$ there exists a sequence of non-amenable groups (even containing non-abelian free subgroups) whose growth rates approach 1 (see [4]).

For a group H, we denote by H' its derived subgroup, that is, $H' = [H, H]$.

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2 Results

Let F_m be a free group of rank m with free basis A. Suppose that R is a normal subgroup of F_m. Assume that there is a homomorphism ϕ from F_m onto an infinite cyclic group whose kernel contains R (that is, F_m/R has the additive group $\mathbb{Z}$ as a homomorphic image). By a we denote a letter from $A^{\pm 1}$ such that

$$\phi(a) = \max\{\phi(x) \mid x \in A^{\pm 1}\}.$$

Clearly $\phi(a) \geqslant 1$.

Throughout the paper, we fix a homomorphism ϕ from F_m onto $\mathbb{Z}$, the letter a described above and the value $C = \phi(a)$. By R we will usually denote a normal subgroup of F_m that is contained in the kernel of ϕ.

A word w over $A^{\pm 1}$ is called *good* whenever it satisfies the following conditions:

(1) w is freely irreducible;

(2) the first letter of w is a;

(3) the last letter of w is not a^{-1};

(4) $\phi(w) > 0$.

Let D_k be the set of all good words of length k and let $d_k = |D_k|$.

Lemma 1. *The number of good words of length* $k \geqslant 4$ *satisfies the following inequality*:

$$d_k \geqslant 4m(m-1)^2(2m-1)^{k-4}. \tag{1}$$

In particular, $\lim_{k\to\infty} d_k^{1/k} = 2m - 1$.

Proof. Let Ω be the set of all freely irreducible words v of length $k - 1$ satisfying $\phi(v) \geqslant 0$. The number of freely irreducible words of length $k - 1$ equals $2m(2m-1)^{k-2}$. At least half of them have non-negative image under ϕ, and so $|\Omega| \geqslant m(2m-1)^{k-2}$.

Let Ω_1 be the subset of Ω that consists of all words whose initial letter is different from a^{-1}. We show that $|\Omega_1| \geqslant ((2m-2)/(2m-1))|\Omega|$. It is sufficient to prove that $|\Omega_1 \cap A^{\pm 1}u| \geqslant ((2m-2)/(2m-1))|\Omega \cap A^{\pm 1}u|$ for any word u of length $k-2$. Suppose that $a^{-1}u$ belongs to Ω. For every letter b one has $\phi(b) \geqslant \phi(a^{-1})$. Therefore $bu \in \Omega_1$ for every letter $b \neq a^{-1}$ if bu is irreducible. There are exactly $2m-2$ ways to choose a letter b with the above properties. Hence $|\Omega_1 \cap A^{\pm 1}u|$ and $|\Omega \cap A^{\pm 1}u|$ have $2m-2$ and $2m-1$ elements, respectively. If $a^{-1}u \notin \Omega$, then the two sets coincide.

Now let Ω_2 denote the subset of Ω_1 that consists of all words whose terminal letter is different from a^{-1}. A similar argument implies that

$$|\Omega_2| \geqslant ((2m-2)/(2m-1))|\Omega_1|.$$

It is obvious that av is good if $v \in \Omega_2$. Therefore the number of good words is at least

$$|\Omega_2| \geqslant \frac{2m-2}{2m-1}|\Omega_1| \geqslant \left(\frac{2m-2}{2m-1}\right)^2 |\Omega| \geqslant 4m(m-1)^2(2m-1)^{k-4}.$$

To every word w in $A^{\pm 1}$ one can uniquely assign a path $p(w)$ in the Cayley graph $\mathscr{C} = \mathscr{C}(F/R, A)$ of the group F/R with A the generating set. This is the path that has label w and starts at the identity. We say that a path p is *self-avoiding* if it never visits the same vertex more than once.

Let $\rho = \rho(R)$ be the length of the shortest non-trivial element in a normal subgroup R of F_m.

Lemma 2. *Let R be a normal subgroup of F_m that is contained in the kernel of a homomorphism ϕ from F_m onto $\mathbb{Z}$. Suppose that $k \geqslant 2$ is chosen in such a way that the following inequality holds*:

$$\rho(R) > Ck(2k-3) + 2k - 2. \tag{2}$$

Then any path in the Cayley graph $\mathscr{C}$ of F_m/R labelled by a word of the form $g_1 g_2 \dots g_t$, where $t \geqslant 1$ and $g_s \in D_k$ for all $1 \leqslant s \leqslant t$, is self-avoiding.

Proof. Suppose that p is not self-avoiding, and consider a minimal subpath q between two equal vertices. Clearly $|q| \geqslant \rho \geqslant k$. Therefore q can be represented as $q = g' g_i \dots g_j g''$, where $g_i, \dots, g_j$ are in D_k, the word g' is a proper suffix of some word in D_k and g'' is a proper prefix of some word in D_k. We have $|g'|, |g''| \leqslant k-1$ so that $|g_i \dots g_j| > Ck(2k-3)$. This implies that $j - i + 1$ (the number of sections that are completely contained in q) is at least $C(2k-3)+1$. Obviously $\phi(g') \geqslant -C(k-1)$ and $\phi(g'') \geqslant -C(k-2)$ (we recall that g'' starts with a if it is non-empty). On the other hand, $\phi(g_s) \geqslant 1$ for all s. Hence

$$\phi(g_i \dots g_j) \geqslant j - i + 1 \geqslant C(2k-3) + 1$$

and so $\phi(g' g_i \dots g_j g'') \geqslant 1$, which is obviously impossible because for every $r \in R$ one has $\phi(r) = 0$.

Theorem 1. *Suppose that R is a normal subgroup of the free group F_m that is contained in the kernel of a homomorphism ϕ from F_m onto $\mathbb{Z}$. Let C be the maximum value of ϕ on the generators and their inverses. Let $\rho = \rho(R)$ be the length of the shortest cyclically irreducible non-empty word in R. If the number $k \geqslant 4$ satisfies the inequality*

$$\rho \geqslant Ck(2k-3) + 2k - 1, \tag{3}$$

then the growth rate of F_m/R' with respect to the natural generators is at least

$$(2m-1) \cdot \left(\frac{4m(m-1)^2}{(2m-1)^4}\right)^{1/k}.$$

Proof. We use the following known fact [2, Lemma 1]: a word w belongs to R' if and only if, for any edge e, the path labelled by w in the Cayley graph of the group F_m/R has the same number of occurrences of e and e^{-1}. Hence distinct self-avoiding paths of length n in the Cayley graph of F_m/R represent distinct elements of the group F_m/R'. Moreover, all of the corresponding paths in the Cayley graph of F_m/R' are geodesic and so these elements have length n in the group F_m/R'.

Suppose that the conditions of the theorem hold. For every n, one can consider the set of all words of the form $g_1 g_2 \dots g_n$, where each g_i belongs to D_k. By Lemma 2 these elements give us distinct self-avoiding paths in the Cayley graph of F_m/R. Hence for any n we have at least d_k^n distinct elements in F_m/R' that have length kn. Therefore the growth rate of F_m/R' is at least $d_k^{1/k}$. It remains to apply Lemma 1.

One can summarize the statement of Theorem 1 as follows: if all relations of F_m/R are long enough, then the growth rate of the group F_m/R' is big enough. Notice that we cannot avoid the assumption that F_m/R projects onto $\mathbb{Z}$. Indeed, for any number ρ, there exists a finite index normal subgroup in F_m all of whose non-trivial elements have length greater than ρ. If R were such a subgroup, then F/R' would be a finite extension of an abelian group and its growth rate would be equal to 1.

Theorem 2. *Let F_m be a free group of rank m with free basis A and let ϕ be a homomorphism from F_m onto $\mathbb{Z}$. Suppose that*

$$\ker \phi \geqslant R_1 \geqslant R_2 \geqslant \cdots \geqslant R_n \geqslant \cdots$$

is a sequence of normal subgroups in F_m with trivial intersection. Then the growth rates of the groups F_m/R_n' approach $2m-1$ as n approaches infinity, that is,

$$\lim_{n \to \infty} \omega(F_m/R_n', A) = 2m - 1.$$

Proof. Since the subgroups R_n have trivial intersection, the lengths of their shortest non-trivial relations approach infinity, that is, $\rho(R_n) \to \infty$ as $n \to \infty$. Let

$$k(n) = [\sqrt{\rho(R_n)/2C}],$$

where C is defined in terms of ϕ as above. Obviously the inequality (3) holds and $k(n) \to \infty$. Theorem 1 implies that the growth rates of the groups F_m/R_n' approach $2m-1$.

Now we show that for every m there exists an amenable group with m generators whose growth rate is arbitrarily close to $2m-1$.

Theorem 3. *For every $m \geqslant 1$ and for every $\varepsilon > 0$, there exists an m-generated amenable group G, which is an extension of an abelian group by a nilpotent group such that the growth rate of G is at least $2m-1-\varepsilon$.*

Proof. It suffices to take the lower central series in the statement of Theorem 2 (that is, $R_1 = F_m'$, $R_{i+1} = [R_i, F_m]$ for all $i \geqslant 1$). The subgroups R_n have trivial intersection and they are contained in F_m' and hence certainly lie in kernels of epimorphisms to $\mathbb{Z}$. The groups $G_n = F_m/R_n'$ are extensions of (free) abelian groups R_n/R_n' by (free) nilpotent groups F_m/R_n and so they are all amenable. Their growth rates approach $2m-1$.

One can take instead the sequence $R_n = F_m^{(n)}$ of iterated derived subgroups (that is, $R_1 = F_m'$, $R_{i+1} = R_i'$ for all $i \geqslant 1$). It is not hard to show that $\rho(R_n)$ grows exponentially. The groups $F_m/R_n' = F_m/R_{n+1}$ are free soluble. Their growth rates approach $2m-1$ very quickly. For instance, the growth rate of the free soluble group of degree 15 with 2 generators is greater than 2.999.

One more application of Theorem 3 can be obtained as follows. The group F_m has countably many finite index normal subgroups and so one can enumerate them as $N_1, N_2, \dots, N_i, \dots$. Let $M_i = N_1 \cap N_2 \cap \dots \cap N_i$ and let $R_i = M_i'$ for all $i \geqslant 1$. Obviously the subgroups M_i (and thus the subgroups R_i) have trivial intersection since F_m is residually finite. As above, all subgroups R_i are contained in F_m' and so in kernels of epimorphisms to $\mathbb{Z}$. Hence the growth rates of the groups $F_m/R_i' = F_m/M_i''$ approach $2m-1$. These groups are extensions of M_i/M_i'' by F_m/M_i, that is, they are finite extensions of (free) metabelian groups.

Therefore there exist m-generated groups with growth rates approaching $2m-1$ in each of the following: (1) the class of extensions of abelian groups by nilpotent groups, and (2) the class of finite extensions of metabelian groups.

Remark. A. Yu. Ol'shanskii suggested the following improvement. Let p be a prime. Since F_m is residually a finite p-group, there is a chain $M_1 \geqslant M_2 \geqslant \cdots$ of normal subgroups with trivial intersection, where each F_m/M_i is a finite p-group. Let $R_i = \ker \phi \cap M_i$. The group F_m/R_i is a subdirect product of $\mathbb{Z}$ and a finite p-group. In particular, it is nilpotent. Moreover, it is also an extension of $\mathbb{Z}$ by a finite p-group and an extension of a finite p-group by $\mathbb{Z}$. So F_m/R_i' will be both abelian-by-nilpotent and metabelian-by-finite. (In fact, the metabelian part is an extension of an abelian group by $\mathbb{Z}$.) Also F_m/R_i' is an extension of a virtually abelian group by $\mathbb{Z}$.

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